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Applications of Incomplete Gamma Functions to the Incomplete Normal Distribution

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ABSTRACT

This paper gives a derivation of a relationship that can be used to estimate the area under a Normal Distribution through the use of Incomplete Gamma Functions.

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1. INTRODUCTION

Problem: A normalized, probability density function (pdf) is given by the function

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \textbf{[1]} \qquad (1)$$

The domain of this function is defined on $x \in (-\infty, \infty)$. If x is a continuous random variable then the probability is defined as the area under the probability density function whose value can be computed by the normalized integral

$$\int_{-\infty}^{\infty} P(x)dx = 1. \textbf{[1]} \qquad (2)$$

If we put equation (1) into equation (2) we find that the integral which needs to be computed is

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1. \textbf{[1]} \quad (3)$$

On the other hand, what if we want to figure out the area for an arbitrary value of x? If we want to do this then we have to compute an incomplete Normal Distribution which is written as

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{0}^{x} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx, for \ x \in [0, \infty). \textbf{[1]} \quad (4)$$

Equation (4) is defined for values of x greater than or equal to zero on the positive domain. In this paper, a derivation will be given through the use of Incomplete Gamma functions to find a general relationship that can be used to give estimates to the value of equation (4) for an arbitrary value of x in the positive domain.

Proof:

Start off with the definition of an incomplete gamma function defined on the complex plane where the real part is greater than zero. Equation (5) is referred to as the Lower Incomplete Gamma Function.

$$\gamma(z, x) = \int_{0}^{x} t^{z-1} e^{-t} dt, Re(z) > 0 \textbf{ [2]} \ (5)$$

Now multiply both sides of (5) by $\frac{1}{\sigma\sqrt{2\pi}}$ to get the following result.

$$\frac{\gamma(z,x)}{\sigma\sqrt{2\pi}} = \frac{1}{\sigma\sqrt{2\pi}} \int_{0}^{x} t^{z-1} e^{-t} dt, Re(z) > 0 \ (6)$$

Using equation (4) as a guide it is clear that the following substitutions can be made and put into equation (6).

$$t = \frac{(x-\mu)^2}{2\sigma^2}, \frac{dt}{dx} = \frac{x-\mu}{\sigma^2} \rightarrow dt = \frac{(x-\mu)dx}{\sigma^2} \ (7)$$

Now put the substitutions from (7) into equation (6) in order to derive equation (8).

$$\frac{\gamma(z,x)}{\sigma\sqrt{2\pi}} = \frac{1}{\sigma\sqrt{2\pi}}\int_0^x \frac{(x-\mu)}{2^{z-1}\sigma^{2z-2}}^{2z-2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\frac{(x-\mu)dx}{\sigma^2}$$

$$\frac{\gamma(z,x)}{\sigma\sqrt{2\pi}} = \frac{1}{2^{z-1}\sigma^{2z}\sigma\sqrt{2\pi}}\int_0^x (x-\mu)^{2z-1} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \qquad (8)$$

The quantity $\frac{1}{2^{z-1}\sigma^{2z}}$ has been moved over the integral sign because it is constant. Now through some algebra we move this quantity over to the left of the equal sign in order to give us the result of equation (9) that will be worked with throughout the rest of this proof.

$$\frac{2^{z-1}\sigma^{2z}\gamma(z,x)}{\sigma\sqrt{2\pi}} = \frac{1}{\sigma\sqrt{2\pi}}\int_0^x (x-\mu)^{2z-1} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \quad (9)$$

Since we are only concerned with the real part of the complex plane we can find some value of z that will eliminate $(x-\mu)^{2z-1}$from the integral. The best way to find that value for z is to set the term in the exponent equal to zero, and solve for z.

$$2z - 1 = 0, 2z = 1, z = \frac{1}{2} \quad (10)$$

Taking the value found for z in equation (10), and putting it back into equation (9) will give a more simplified result that is shown for equation (11).

$$\frac{\gamma\left(\frac{1}{2},x\right)}{2\sqrt{\pi}} = \frac{1}{\sigma\sqrt{2\pi}}\int_0^x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \quad (11)$$

What is special about equation (11) is the quantity of $\gamma\left(\frac{1}{2},x\right)$ because there are specific identities that are associated with this specific lower incomplete gamma function. The general identities that will be used are as follows:

$$\gamma(z,x) + \Gamma(z,x) = \Gamma(z) \;\textbf{[2]} \quad (12)$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \;\textbf{[2]} \quad (13)$$

Equation (12) has also has the quantity, $\Gamma(z,x)$ which is defined as the incomplete upper gamma function. When the value of ½ is put into equation (11) for z this means that equation (12) turns into equation (14)

$$\gamma\left(\frac{1}{2},x\right) + \Gamma\left(\frac{1}{2},x\right) = \Gamma\left(\frac{1}{2}\right) \;\textbf{[2]} \quad (14)$$

While equation (13) turns into equation (15) which is the following result

$$\Gamma^2\left(\frac{1}{2}\right) = \frac{\pi}{\sin(\frac{\pi}{2})} = \pi\,. \quad (15)$$

Taking the square root of equation (15) gives the useful result for equation (16).

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad (16)$$

Now put equation (16) into equation (14) to obtain the result that will be equation (17).

$$\gamma\left(\frac{1}{2},x\right) = \sqrt{\pi} - \Gamma\left(\frac{1}{2},x\right) \quad (17)$$

Two more identities that will help deal with equation (17) are as follows in equation (18) and equation (19).

$$\Gamma\left(\frac{1}{2},x\right) = \sqrt{\pi}erfc(\sqrt{x}) \textbf{ [2]} \quad (18)$$

$$\mathrm{erf}(\sqrt{x}) + erfc(\sqrt{x}) = 1 \textbf{ [2]} \quad (19)$$

Now take equation (19) and solve it for $erfc(\sqrt{x})$ to get equation (20). The result for this is

$$erfc(\sqrt{x}) = 1 - \mathrm{erf}(\sqrt{x}). \quad (20)$$

Taking equation (20), and putting it into equation (18) gives equation (21) which is a very useful result.

$$\Gamma\left(\frac{1}{2},x\right) = \sqrt{\pi}(1 - \mathrm{erf}(\sqrt{x})) \quad (21)$$

Now put equation (21) into equation (17) to give a value for that can be used in equation (11). This result will be designated as equation (22).

$$\gamma\left(\frac{1}{2},x\right) = \sqrt{\pi} - \Gamma\left(\frac{1}{2},x\right) = \sqrt{\pi} - \sqrt{\pi}\left(1 - \mathrm{erf}(\sqrt{x})\right) = \sqrt{\pi}\,\mathrm{erf}(\sqrt{x}) \quad (22)$$

Finally, putting equation (22) into equation (11) gives equation (24), the general relationship that can be used to give estimates to the value of equation (4) for an arbitrary value of x in the positive domain.

$$\frac{1}{\sigma\sqrt{2\pi}}\int_0^x e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \frac{\mathrm{erf}(\sqrt{x})}{2} \quad (23)$$

This gives a way of calculating the area under one-half of a normalized Gaussian. The function, erf(u) is what is known as the error function and has the following generalized form as stated in equation (24).

$$For\ u \ll 1, \mathrm{erf}(u) = \frac{1}{\sqrt{\pi}}e^{-u^2}\sum_{n=0}^{\infty}\frac{(2u)^{2n+1}}{(2n+1)!!} \textbf{ [3]}$$

$$For\ u \gg 1, \mathrm{erf}(u) \sim 1 - \frac{e^{u^2}}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n}u^{-(2n+1)} \textbf{ [3]} \quad (24)$$

When setting u equal to the square root of x, and equating equation (23) to equation (24) we get the following results for the equations in (25).

$$For\ \sqrt{x} \ll 1, \quad \frac{1}{\sigma\sqrt{2\pi}}\int_0^x e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \frac{1}{2}[\frac{1}{\sqrt{\pi}}e^{-x}\sum_{n=0}^{\infty}\frac{\left(2\sqrt{x}\right)^{2n+1}}{(2n+1)!!}]$$

$$For\ \sqrt{x} \gg 1, \ \frac{1}{\sigma\sqrt{2\pi}}\int_0^x e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \frac{1}{2}[1 - \frac{e^x}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n}x^{-\frac{2n+1}{2}}] \ (25)$$

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2. APPLICATION

In looking at the equations in (25) as x approaches 0, and x approaches ∞ we get the following results.

$$For\ \sqrt{x} \ll 1, \qquad \frac{1}{\sigma\sqrt{2\pi}}\int_0^0 e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \lim_{x\to 0}\frac{1}{2}[\frac{1}{\sqrt{\pi}}e^{-x}\sum_{n=0}^{\infty}\frac{\left(2\sqrt{x}\right)^{2n+1}}{(2n+1)!!}] = 0$$

$$For\ \sqrt{x} \gg 1,\ \frac{1}{\sigma\sqrt{2\pi}}\int_0^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \lim_{x\to\infty}\frac{1}{2}\left[1-\frac{e^x}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n}x^{-\frac{2n+1}{2}}\right] = \frac{1}{2}\ (26)$$

While the conclusion for the first equation in (26) is obvious, there is another interesting result that is recovered when looking at the second equation in (26). Since the Normal Distribution is an even function then we can use equation (27) to find the total area under the normal distribution.

$If\ f(x) is\ an\ even\ function\ on\ the\ interval\ [-l, l] then\ \int_{-l}^{l} f(x)dx = 2\int_0^l f(x)dx$ **[4]** (27)

If we take the limits of integration to their infinite limits on the closed interval$[-\infty, \infty]$, and apply equation (27) to equation (26) we get the area under a Normal Distribution.

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = 2*\frac{1}{\sigma\sqrt{2\pi}}\int_0^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = 1\ (28)$$

Finally, if we set the mean, μ equal to 0, and the standard deviation, σ equal to one we get the normalized form of equation (3). Which will yield the entire area under a bell curve as displayed in equation (29).

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx = 1$$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx = \sqrt{2\pi}\ \ (29)$$

3. CONCLUSION

Through the Lower Incomplete Gamma Function we have shown that it can be used to derive a result that can be used to calculate the area under a Gaussian integral from either 0 to some value of x, or from 0 out to some very large value.

$$For\ \sqrt{x} \ll 1, \qquad \frac{1}{\sigma\sqrt{2\pi}}\int_0^x e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \frac{1}{2}[\frac{1}{\sqrt{\pi}}e^{-x}\sum_{n=0}^{\infty}\frac{\left(2\sqrt{x}\right)^{2n+1}}{(2n+1)!!}]$$

$$For\ \sqrt{x} \gg 1, \qquad \frac{1}{\sigma\sqrt{2\pi}}\int_0^x e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \frac{1}{2}[1-\frac{e^x}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n}x^{-\frac{2n+1}{2}}]$$

(25)

Finally, we showed through application that the result given in equation (25), when it has the limiting cases from equation (26), along with applied to it give an expected result given in equation (29).

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx = 1$$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx = \sqrt{2\pi} \quad (29)$$

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