Summer 2015

A Short Note On Sums of Powers of Reciprocals of Polygonal Numbers

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January 3, 2017

Abstract

This paper presents the summation of powers of reciprocals of polygonal numbers. Several summation formulas of the reciprocals of generalized polygonal numbers are presented as examples of specific cases in this paper.

Acknowledgement

The authors would like to thank the Faculty Development Committee of DePauw University for funding this summer research project during summer 2015.

1 Introduction

Polygonal numbers are numbers that represent dots that are arranged in the shape of a geometric regular polygon. Generalized polygonal numbers of rank r are numbers of the form [1][2][16][17]

$$P_k^r = \frac{k[(r-2)k-(r-4)]}{2} \tag{1}$$

where P_k^r is k-th r-gonal number. For example, P_3^r gives the triangular number, while P_5^r gives the pentagonal number.

Cook and Bacon observed in their paper [6], that specific examples of sums of reciprocals of various polygonal numbers are worth further study and thus motivates this paper. Before we present the summation of reciprocal formulas, we note the following interesting lemma about polygonal numbers.

Lemma 1 *The sum of the first n polygonal numbers is given by [6]*

$$\sum_{k=1}^{n} P_k^r = \frac{n(n+1)[(r-2)n+5-r]}{6} (1.2)$$

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PROOF. From Equation 1, we can see that,

$$\sum_{k=1}^{n} P_k^r = \frac{r-2}{2}\sum_{k=1}^{n} k^2 - \frac{r-4}{2}\sum_{k=1}^{n} k = \frac{n(n+1)[(r-2)(2n+1)-3(r-4)]}{12} = \frac{n(n+1)[(r-2)n+5-r]}{6} \quad (2)$$

□

Equation 2 gives the results for $r = 10, 11, 12, ..., 15$ and we state it here, for the sake of completeness, as the following corollary. (For results relating to $r = 1, \ldots 9$, refer to [6]).

Corollary 2 *For* $r = 10, 11, 12, ..., 15$*,*

$$\sum_{k=1}^{n} P_k^{10} = \frac{(n+1)n(8n-5)}{6}$$

$$\sum_{k=1}^{n} P_k^{11} = \frac{(n+1)n(9n-6)}{6}$$

$$\sum_{k=1}^{n} P_k^{12} = \frac{(n+1)n(10n-7)}{6}$$

$$\sum_{k=1}^{n} P_k^{13} = \frac{(n+1)n(11n-8)}{6}$$

$$\sum_{k=1}^{n} P_k^{14} = \frac{(n+1)n(12n-9)}{6}$$

$$\sum_{k=1}^{n} P_k^{15} = \frac{(n+1)n(13n-10)}{6}$$

We now divide the remaining paper into the following sections. The next section will deal with preliminary definitions while sections three and four deal with some results regarding finite and infinite sums of reciprocals of powers of polygonal numbers respectively.

2 Preliminary Definitions

For other notations and terminology not defined here, see Abramowitz and Stegun's Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables [1] and Gradshteyn and Ryzhik's Table of Integrals, Series, and Products [8].

Definition 1 [1],[8] The Harmonic number H_n is the sum of the reciprocals of the first n natural numbers and is given by,

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k} \quad (3)$$

Definition 2 [1],[8] The generalized Harmonic number of order n, m is given by,

$$H_{n,m}=\sum_{k=1}^{n}\frac{1}{k^m} \tag{4}$$

Definition 3 [1],[8] The generalized Harmonic function $H_n^{(s)}(z)$ is defined by [5][4][11][14]

$$H_n^{(s)}(z)=\sum_{j=1}^{n}\frac{1}{(j+z)^s} \tag{5}$$

, where $n\in\mathbb{N}$,$s\in\mathbb{C}|\mathbb{Z}^-$; and $\mathbb{Z}^- := \{-1,-2,-3,...\}$.

Definition 4 [1],[8] The error function (denoted by $erf(x)$) which is typically encountered when integrating the normal distribution, is defined by

$$erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \tag{6}$$

3 Finite Sums of Reciprocal of Powers of Polygonal Numbers

We now consider the finite sum of the reciprocals of the powers of polygonal numbers.

Lemma 3 *Let P_k^r denote the k^{th} polygonal number of rank r. Then* $\sum_{k=1}^{n}\frac{1}{P_k^r}=-\frac{2}{\beta(r-2)}H_{n,1}+\frac{2}{\beta(r-2)}H_n^{(1)}(-\frac{r-4}{r-2})$.

Proof.

$$\sum_{k=1}^{n}\frac{1}{P_k^r}=\sum_{k=1}^{n}\frac{2}{k[(r-2)k-(r-4)]}$$

Let $\beta=\frac{r-4}{r-2}$, then

$$\sum_{k=1}^{n}\frac{1}{P_k^r}=\sum_{k=1}^{n}\frac{\frac{2}{r-2}}{k(k-\beta)}$$

$$=\sum_{k=1}^{n}(-\frac{2}{k\beta(r-2)}+\frac{2}{(k-\beta)\beta(r-2)})$$

$$=-\frac{2}{\beta(r-2)}\sum_{k=1}^{n}(\frac{1}{k})+\frac{2}{\beta(r-2)}\sum_{k=1}^{n}(\frac{1}{(k-\beta})$$

By the definition of harmonic number (Definitions 2 and 3), we know that

$$\sum_{k=1}^{n}\frac{1}{P_k^r}=-\frac{2}{\beta(r-2)}H_{n,1}+\frac{2}{\beta(r-2)}H_n^{(1)}(-\beta)$$

$$=-\frac{2}{\beta(r-2)}H_{n,1}+\frac{2}{\beta(r-2)}H_n^{(1)}(-\frac{r-4}{r-2})$$

□

We must mention that our proof above though slightly different, does indeed verify the sum $\sum_{k=1}^{n}\frac{1}{P_k^r}$ that has been proved in [10].

Lemma 4 *Let P_k^r denote the k^{th} polygonal number of rank r. Then* $\sum_{k=1}^{n}(\frac{1}{P_k^r})^2 = \frac{8}{\beta^3(r-2)^2}H_{n,1} - \frac{8}{\beta^3(r-2)^2}H_n^{(1)}(-\frac{r-4}{r-2}) + \frac{4}{\beta^2(r-2)^2}H_{n,2} + \frac{4}{\beta^2(r-2)^2}H_n^{(2)}(-\frac{r-4}{r-2})$ *for $r \neq 4$, where* $\beta = \frac{r-4}{r-2}$

PROOF.

$$\sum_{k=1}^{n}(\frac{1}{P_k^r})^2 = \sum_{k=1}^{n}\frac{4}{k^2[(r-2)k-(r-4)]^2}$$

By a partial fraction expansion, the sum becomes,

$$\sum_{k=1}^{n}(\frac{1}{P_k^r})^2 = \sum_{k=1}^{n}\frac{\frac{4}{(r-2)^2}}{k^2(k-\beta)^2}$$

$$= \sum_{k=1}^{n}(\frac{8}{k\beta^3(r-2)^2} - \frac{8}{(k-\beta)\beta^3(r-2)^2} + \frac{4}{k^2\beta^2(r-2)^2} + \frac{4}{(k-\beta)^2\beta^2(r-2)^2})$$

$$= \frac{8}{\beta^3(r-2)^2}\sum_{k=1}^{n}(\frac{1}{k}) - \frac{8}{\beta^3(r-2)^2}\sum_{k=1}^{n}(\frac{1}{(k-\beta)}) + \frac{4}{\beta^2(r-2)^2}\sum_{k=1}^{n}(\frac{1}{k^2}) + \frac{4}{\beta^2(r-2)^2}\sum_{k=1}^{n}(\frac{1}{(k-\beta)^2})$$

By definitions (4) and (5),

$$\sum_{k=1}^{n}(\frac{1}{P_k^r})^2 = \frac{8}{\beta^3(r-2)^2}H_{n,1} - \frac{8}{\beta^3(r-2)^2}H_n^{(1)}(-\frac{r-4}{r-2}) + \frac{4}{\beta^2(r-2)^2}H_{n,2} + \frac{4}{\beta^2(r-2)^2}H_n^{(2)}(-\frac{r-4}{r-2})$$

□

Theorem 5 *Let P_k^r denote the k^{th} polygonal number of rank r. Then, for $r \neq 4$, $m \geq 2$,*

$$\sum_{k=1}^{n}(\frac{1}{P_k^r})^m = \sum_{i=1}^{m-1}[(-1)^m C_i^m \frac{(\frac{1}{r-2})^m}{(\frac{r-4}{r-2})^{m+i-1}}H_{n,m-i+1}] + (-1)^m C_m^m \frac{(\frac{1}{r-2})^m}{(\frac{r-4}{r-2})^{m+i-1}}H_{n,m-i+1}$$

$$+\sum_{i=1}^{m-1}[(-1)^{i-1}(-1)^m C_i^m \frac{(\frac{1}{r-2})^m}{(\frac{r-4}{r-2})^{m+i-1}}H_n^{(m-i+1)}(-\frac{r-4}{r-2})] + (-1)^{m-1}C_m^m \frac{(\frac{1}{r-2})^m}{(\frac{r-4}{r-2})^{m+i-1}}H_n^{(m-i+1)}(-\frac{r-4}{r-2})$$

.

PROOF. Let P_k^r denote the k^{th} polygonal number of rank r. From (1) we can see that

$$\frac{1}{P_k^r} = \frac{2}{k[(r-2)k-(r-4)]},$$

so for the m^{th} power of P_k^r, we have

$$(\frac{1}{P_k^r})^m = \frac{2^m}{(r-2)^m k^m (k-\frac{r-4}{r-2})^m}.$$

By a tedious partial fraction expansion, we can see that

$$\sum_{k=1}^{n}(\frac{1}{P_k^r})^m = \sum_{i=1}^{m-1}[(-1)^m C_i^m \frac{(\frac{1}{r-2})^m}{(\frac{r-4}{r-2})^{m+i-1}}\sum_{k=1}^{n}\frac{1}{k^{m-i+1}}] + (-1)^m C_m^m \frac{(\frac{1}{r-2})^m}{(\frac{r-4}{r-2})^{m+i-1}}\sum_{k=1}^{n}\frac{1}{k}$$

$$+\sum_{i=1}^{m-1}[(-1)^{i-1}(-1)^m C_i^m \frac{(\frac{1}{r-2})^m}{(\frac{r-4}{r-2})^{m+i-1}} \sum_{k=1}^{n} \frac{1}{(k-\frac{r-4}{r-2})^{m-i+1}}] + (-1)^{m-1} C_m^m \frac{(\frac{1}{r-2})^m}{(\frac{r-4}{r-2})^{m+i-1}} \sum_{k=1}^{n} \frac{1}{k-\frac{r-4}{r-2}}$$

$$= \sum_{i=1}^{m-1}[(-1)^m C_i^m \frac{(\frac{1}{r-2})^m}{(\frac{r-4}{r-2})^{m+i-1}} H_{n,m-i+1}] + (-1)^m C_m^m \frac{(\frac{1}{r-2})^m}{(\frac{r-4}{r-2})^{m+i-1}} H_{n,m-i+1}$$

$$+\sum_{i=1}^{m-1}[(-1)^{i-1}(-1)^m C_i^m \frac{(\frac{1}{r-2})^m}{(\frac{r-4}{r-2})^{m+i-1}} H_n^{(m-i+1)}(-\frac{r-4}{r-2})] + (-1)^{m-1} C_m^m \frac{(\frac{1}{r-2})^m}{(\frac{r-4}{r-2})^{m+i-1}} H_n^{(m-i+1)}(-\frac{r-4}{r-2})$$

$$C_i^m = 2C_{i-1}^m + C_i^{m-1}$$

$$C_m^m = 2C_m^{m-1}$$

(In order to generalize the above formula, we used Mathematica to obtain the partial fraction expansions for $m = 2, \ldots, 10$ - see appendix for the mathematica results), □

4 Infinite Sums of Reciprocal of Powers of Polygonal Numbers

From results in our previous section, we now validate a formula for $\sum_{k=1}^{\infty} \frac{1}{P_k^r}$, which has already been found differently in [15].

Lemma 6 *Let P_k^r denote the k^{th} polygonal number of rank r. Then $\sum_{k=1}^{\infty} \frac{1}{P_k^r} = \frac{2}{r-4} H_{-\beta}$, where $\beta = \frac{r-4}{r-2}$, $r \neq 4$.*

Proof.

$$\sum_{k=1}^{\infty} \frac{1}{P_k^r} = \sum_{k=1}^{\infty} \frac{2}{k[(r-2)k-(r-4)]}$$

Let $\beta = \frac{r-4}{r-2}$, *then*

$$\sum_{k=1}^{\infty} \frac{1}{P_k^r} = \frac{2}{r-2} \sum_{k=1}^{\infty} \frac{1}{k(k-\beta)}$$

$$= \frac{2}{r-2} \sum_{k=1}^{\infty} \frac{1}{\beta}(\frac{1}{k-\beta} - \frac{1}{k})$$

$$= \frac{2}{r-4} \sum_{k=1}^{\infty}(\frac{1}{k-\beta} - \frac{1}{k})$$

Since, for $A > 0$, $\frac{1}{A} = \int_0^{\infty} e^{-Ax} dx$ [7], we get,

$$\sum_{k=1}^{\infty} \frac{1}{P_k^r} = \frac{2}{r-4} \sum_{k=1}^{\infty} \int_0^{\infty} e^{-(k-\beta)x} - e^{-kx} dx$$

$$= \frac{2}{r-4} \lim_{N \to +\infty} \sum_{k=1}^{N} \int_0^{\infty} e^{-kx}(e^{\beta x} - 1) dx$$

By the Monotone Convergence Theorem [13], the above sum becomes,

$$= \frac{2}{r-4} \lim_{N\to+\infty} \int_0^\infty \frac{(1-e^{-Nx})e^{-x}}{1-e^{-x}}(e^{\beta x}-1)dx$$

which then becomes (after taking the limit as $N \to \infty$),

$$\sum_{k=1}^{\infty} \frac{1}{P_k^r} = \frac{2}{r-4} \int_0^\infty \frac{e^{-x}}{1-e^{-x}}(e^{\beta x}-1)dx$$

Let $t = e^{-x}$, then

$$\sum_{k=1}^{\infty} \frac{1}{P_k^r} = \frac{2}{r-4} \int_0^1 \frac{t^{-\beta}-1}{1-t}dt$$

By using Mathematica, we find that $H_{-\beta} = \int_0^1 \frac{t^{-\beta}-1}{1-t}dt$,

$$\sum_{k=1}^{\infty} \frac{1}{P_k^r} = \frac{2}{r-4} H_{-\beta}$$

□

Lemma 7 *Let $f(k) = e^{-k^2 x} + e^{-(k-\beta)^2 x}$, $x > 0$. Then $f(k) > 0$, when $k \in [1, \infty)$*

Proof. The exponential function $e^n > 0$. When $n = -k^2 x$ or $n = -(k-\beta)^2 x$, e^n is still greater than zero. Therefore, $f(k) > 0$

□

Lemma 8 *Let $f(k) = e^{-k^2 x} + e^{-(k-\beta)^2 x}$. Then $f(k)$ is a decreasing function on $[1, \infty)$, when $x > 0$ and $\beta = \frac{r-4}{r-2}, r \geq 3$.*

Proof. Let $k_1 > k_2 \geq 1$, then we have $k_1^2 > k_2^2$. Then $e^{-k_1^2 x} < e^{-k_2^2 x}$, when $x > 0$. Since $\beta = \frac{r-4}{r-2}$ and $r \geq 3$, we know that $\beta \geq -1$. Then $e^{-(k_1-\beta)^2 x} < e^{-(k_2-\beta)^2 x}$, when $x > 0$. Therefore,

$$e^{-k_1^2 x} + e^{-(k_1-\beta)^2 x} < e^{-k_2^2 x} + e^{-(k_2-\beta)^2 x}.$$

When $k_1 > k_2 \geq 1$, The$f(k_1) < f(k_2)$, so $f(k)$ is a decreasing function.

□

Lemma 9 *Let $f(k) = e^{-k^2 x} + e^{-(k-\beta)^2 x}$, and I_n be a sequence where $I_n = \int_1^N f(k)dk$ and $x > 0$. Then I_n is bounded.*

Proof.

$$I_n = \int_1^N f(k)dk = \int_1^N e^{-k^2 x} + e^{-(k-\beta)^2 x} dk$$

$$= \frac{\sqrt{\pi}(-erf(\sqrt{x}) + erf(N\sqrt{x}))}{2\sqrt{x}} + \frac{\sqrt{\pi}(erf((N-\beta)\sqrt{x}) + erf((\beta-1)\sqrt{x}))}{2\sqrt{x}}$$

,where $erf(x)$ is the error function by Definition 4.

$$I_n = \frac{\sqrt{\pi}}{2\sqrt{x}}(-erf(\sqrt{x}) + erf(N\sqrt{x}) + erf((N-\beta)\sqrt{x}) + erf((\beta-1)\sqrt{x})).$$

For error function, we have $erf(z) = 1$ when $z \to \infty$, and $erf(z) = -1$ when $z \to -\infty$, and the range of error function is $(-1, 1)$, then we have

$$erf(\sqrt{x}) < 1$$

$$erf(N\sqrt{x}) = 1$$

$$erf((N-\beta)\sqrt{x}) = 1$$

$$erf((\beta-1)\sqrt{x}) < 1$$

$$(-erf(\sqrt{x}) + erf(N\sqrt{x}) + erf((N-\beta)\sqrt{x}) + erf((\beta-1)\sqrt{x})) < 4$$

$$\frac{\sqrt{\pi}}{2\sqrt{x}}(-erf(\sqrt{x}) + erf(N\sqrt{x}) + erf((N-\beta)\sqrt{x}) + erf((\beta-1)\sqrt{x})) < \frac{2\sqrt{\pi}}{\sqrt{x}}$$

Therefore, I_n is bounded above by 4, when $N \to \infty$. □

Lemma 10 *Let P_k^r denote the k^{th} polygonal number of rank r. Then we claim that*

$$\sum_{k=1}^{\infty}(\frac{1}{P_k^r})^2 =$$

$$= \frac{4}{(r-4)^2}(\int_0^{\infty}\frac{2e^{-x}}{1-e^{-x}}(1-e^{\beta^2 x})dx + \int_0^{\infty}\frac{1}{2}(1+EllipticTheta[3,0,e^{-x}])dx + \lim_{N\to+\infty}\sum_{k=1}^{N}\int_0^{\infty}e^{-(k-\beta)^2 x}dx)$$

if

$$\lim_{N\to+\infty}\int_0^{\infty}2\frac{(1-e^{-N\beta x})e^{-x}}{1-e^{-x}}(1-e^{\beta^2 x})dx + \lim_{N\to+\infty}\sum_{k=1}^{N}\int_0^{\infty}e^{-k^2 x}dx + \lim_{N\to+\infty}\sum_{k=1}^{N}\int_0^{\infty}e^{-(k-\beta)^2 x}dx$$

is convergent.

PROOF.

$$\sum_{k=1}^{\infty}(\frac{1}{P_k^r})^2 = \sum_{k=1}^{\infty}\frac{4}{k^2[(r-2)k-(r-4)]^2}$$

Let $\beta = \frac{r-4}{r-2}$, *then*

$$\sum_{k=1}^{\infty}(\frac{1}{P_k^r})^2 = \frac{4}{(r-2)^2}\sum_{k=1}^{\infty}\frac{1}{k^2(k-\beta)^2}$$

$$= \frac{4}{(r-4)^2}\sum_{k=1}^{\infty}(\frac{2}{k\beta} - \frac{2}{(k-\beta)\beta} + \frac{1}{k^2} + \frac{1}{(k-\beta)^2})$$

Since $\frac{1}{A} = \int_0^{\infty}e^{-Ax}dx$ for $A > 0$,

$$\sum_{k=1}^{\infty}(\frac{1}{P_k^r})^2 = \frac{4}{(r-4)^2}\sum_{k=1}^{\infty}\int_0^{\infty}(2e^{-k\beta x} - 2e^{-(k-\beta)\beta x} + e^{-k^2 x} + e^{-(k-\beta)^2 x})dx$$

$$= \frac{4}{(r-4)^2} \lim_{N\to+\infty} \sum_{k=1}^{N} \int_0^\infty (2e^{-k\beta x}(1-e^{\beta^2 x}) + e^{-k^2 x} + e^{-(k-\beta)^2 x})dx$$

From the Theorem of Linearity of Infinite Series [12], we have

$$\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n + b_n)$$

$$\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n - b_n)$$

when $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent. For $\sum_{k=1}^{\infty}(\frac{1}{P_k^r})^2$, if $\sum_{k=1}^{N}(2e^{-k\beta x}(1-e^{\beta^2 x})$, $\sum_{k=1}^{N}(e^{-k^2 x})$, $\sum_{k=1}^{N}(e^{-(k-\beta)^2 x})$ converge, then we have

$$\sum_{k=1}^{\infty}(\frac{1}{P_k^r})^2 = \frac{4}{(r-4)^2}(\lim_{N\to+\infty}\int_0^\infty 2\frac{(1-e^{-N\beta x})e^{-x}}{1-e^{-x}}(1-e^{\beta^2 x})dx + \lim_{N\to+\infty}\sum_{k=1}^{N}\int_0^\infty e^{-k^2 x}dx + \lim_{N\to+\infty}\sum_{k=1}^{N}\int_0^\infty e^{-(k-\beta)^2 x}dx)$$

$$= \frac{4}{(r-4)^2}(\int_0^\infty \frac{2e^{-x}}{1-e^{-x}}(1-e^{\beta^2 x})dx + \int_0^\infty \frac{1}{2}(1 + EllipticTheta[3,0,e^{-x}])dx + \lim_{N\to+\infty}\sum_{k=1}^{N}\int_0^\infty e^{-(k-\beta)^2 x}dx)$$

□

Conjecture 11 *Let P_k^r denote the k^{th} polygonal number of rank r. Then*

$$\sum_{k=1}^{\infty}(\frac{1}{P_k^r})^\alpha,$$

$\alpha \geq 3$, $\alpha \in \mathbb{N}$ has a closed form similar to the form for $\sum_{k=1}^{\infty}(\frac{1}{P_k^r})^2$ in the previous Lemma 10,

Appendix A Mathemetica

Apart[4/(k^2((r − 2)k − (r − 4))^2), k]

$$\frac{4}{k^2(-4+r)^2} + \frac{8(-2+r)}{k(-4+r)^3} + \frac{4(-2+r)^2}{(-4+r)^2(4-2k-r+kr)^2} - \frac{8(4-4r+r^2)}{(-4+r)^3(4-2k-r+kr)}$$

Apart[8/(k^3((r − 2)k − (r − 4))^3), k]

$$-\frac{8}{k^3(-4+r)^3} - \frac{24(-2+r)}{k^2(-4+r)^4} - \frac{48(-2+r)^2}{k(-4+r)^5} + \frac{8(-2+r)^3}{(-4+r)^3(4-2k-r+kr)^3} + \frac{48(-2+r)^3}{(-4+r)^5(4-2k-r+kr)} - \frac{24(-8+12r-6r^2+r^3)}{(-4+r)^4(4-2k-r+kr)^2}$$

Apart[16/(k^4((r − 2)k − (r − 4))^4), k]

$$\frac{16}{k^4(-4+r)^4} + \frac{64(-2+r)}{k^3(-4+r)^5} + \frac{160(-2+r)^2}{k^2(-4+r)^6} + \frac{320(-2+r)^3}{k(-4+r)^7} + \frac{16(-2+r)^4}{(-4+r)^4(4-2k-r+kr)^4} - \frac{64(-2+r)^4}{(-4+r)^5(4-2k-r+kr)^3} + \frac{160(-2+r)^4}{(-4+r)^6(4-2k-r+kr)^2} - \frac{320(16-32r+24r^2-8r^3+r^4)}{(-4+r)^7(4-2k-r+kr)}$$

Apart[32/(k^5((r − 2)k − (r − 4))^5), k]

$$\frac{16}{k^4(-4+r)^4} + \frac{64(-2+r)}{k^3(-4+r)^5} + \frac{160(-2+r)^2}{k^2(-4+r)^6} + \frac{320(-2+r)^3}{k(-4+r)^7} + \frac{16(-2+r)^4}{(-4+r)^4(4-2k-r+kr)^4} - \frac{64(-2+r)^4}{(-4+r)^5(4-2k-r+kr)^3} + \frac{160(-2+r)^4}{(-4+r)^6(4-2k-r+kr)^2} - \frac{320(16-32r+24r^2-8r^3+r^4)}{(-4+r)^7(4-2k-r+kr)}$$

Apart[64/(k^6((r − 2)k − (r − 4))^6), k]

$$\frac{64}{k^6(-4+r)^6}+\frac{384(-2+r)}{k^5(-4+r)^7}+\frac{3584(-2+r)^3}{k^3(-4+r)^9}+\frac{8064(-2+r)^4}{k^2(-4+r)^{10}}+\frac{16128(-2+r)^5}{k(-4+r)^{11}}+\frac{64(-2+r)^6}{(-4+r)^6(4-2k-r+kr)^6}-\frac{384(-2+r)^6}{(-4+r)^7(4-2k-r+kr)^5}+$$
$$\frac{1344(-2+r)^6}{(-4+r)^8(4-2k-r+kr)^4}+\frac{1344\left(4-4r+r^2\right)}{k^4(-4+r)^8}-\frac{3584\left(64-192r+240r^2-160r^3+60r^4-12r^5+r^6\right)}{(-4+r)^9(4-2k-r+kr)^3}+\frac{8064\left(64-192r+240r^2-160r^3+60r^4-12r^5+r^6\right)}{(-4+r)^{10}(4-2k-r+kr)^2}-$$
$$\frac{16128\left(64-192r+240r^2-160r^3+60r^4-12r^5+r^6\right)}{(-4+r)^{11}(4-2k-r+kr)}$$

Apart[128/(k^7((r − 2)k − (r − 4))^7), k]

$$-\frac{128}{k^7(-4+r)^7}-\frac{896(-2+r)}{k^6(-4+r)^8}-\frac{10752(-2+r)^3}{k^4(-4+r)^{10}}-\frac{26880(-2+r)^4}{k^3(-4+r)^{11}}-\frac{59136(-2+r)^5}{k^2(-4+r)^{12}}-\frac{118272(-2+r)^6}{k(-4+r)^{13}}+\frac{128(-2+r)^7}{(-4+r)^7(4-2k-r+kr)^7}-$$
$$\frac{896(-2+r)^7}{(-4+r)^8(4-2k-r+kr)^6}+\frac{3584(-2+r)^7}{(-4+r)^9(4-2k-r+kr)^5}-\frac{10752(-2+r)^7}{(-4+r)^{10}(4-2k-r+kr)^4}-\frac{3584\left(4-4r+r^2\right)}{k^5(-4+r)^9}+\frac{26880\left(-128+448r-672r^2+560r^3-280r^4+84r^5-14r^6+\right.}{(-4+r)^{11}(4-2k-r+kr)^3}$$
$$\frac{59136\left(-128+448r-672r^2+560r^3-280r^4+84r^5-14r^6+r^7\right)}{(-4+r)^{12}(4-2k-r+kr)^2}+\frac{118272\left(-128+448r-672r^2+560r^3-280r^4+84r^5-14r^6+r^7\right)}{(-4+r)^{13}(4-2k-r+kr)}$$

Apart[256/(k^8((r − 2)k − (r − 4))^8), k]

$$\frac{256}{k^8(-4+r)^8}+\frac{2048(-2+r)}{k^7(-4+r)^9}+\frac{30720(-2+r)^3}{k^5(-4+r)^{11}}+\frac{84480(-2+r)^4}{k^4(-4+r)^{12}}+\frac{202752(-2+r)^5}{k^3(-4+r)^{13}}+\frac{439296(-2+r)^6}{k^2(-4+r)^{14}}+\frac{878592(-2+r)^7}{k(-4+r)^{15}}+\frac{256(-2+r)^8}{(-4+r)^8(4-2k-r+kr)^8}-$$
$$\frac{2048(-2+r)^8}{(-4+r)^9(4-2k-r+kr)^7}+\frac{9216(-2+r)^8}{(-4+r)^{10}(4-2k-r+kr)^6}-\frac{30720(-2+r)^8}{(-4+r)^{11}(4-2k-r+kr)^5}+\frac{84480(-2+r)^8}{(-4+r)^{12}(4-2k-r+kr)^4}+\frac{9216\left(4-4r+r^2\right)}{k^6(-4+r)^{10}}-$$
$$\frac{202752\left(256-1024r+1792r^2-1792r^3+1120r^4-448r^5+112r^6-16r^7+r^8\right)}{(-4+r)^{13}(4-2k-r+kr)^3}+\frac{439296\left(256-1024r+1792r^2-1792r^3+1120r^4-448r^5+112r^6-16r^7+r^8\right)}{(-4+r)^{14}(4-2k-r+kr)^2}-$$
$$\frac{878592\left(256-1024r+1792r^2-1792r^3+1120r^4-448r^5+112r^6-16r^7+r^8\right)}{(-4+r)^{15}(4-2k-r+kr)}$$

Apart[512/(k^9((r − 2)k − (r − 4))^9), k]

$$-\frac{512}{k^9(-4+r)^9}-\frac{4608(-2+r)}{k^8(-4+r)^{10}}-\frac{253440(-2+r)^4}{k^5(-4+r)^{13}}-\frac{658944(-2+r)^5}{k^4(-4+r)^{14}}-\frac{1537536(-2+r)^6}{k^3(-4+r)^{15}}-\frac{3294720(-2+r)^7}{k^2(-4+r)^{16}}-\frac{6589440(-2+}{k(-4+r)^{17}}$$
$$+\frac{512(-2+r)^9}{(-4+r)^9(4-2k-r+kr)^9}-\frac{4608(-2+r)^9}{(-4+r)^{10}(4-2k-r+kr)^8}+\frac{23040(-2+r)^9}{(-4+r)^{11}(4-2k-r+kr)^7}-\frac{84480(-2+r)^9}{(-4+r)^{12}(4-2k-r+kr)^6}$$

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